Worksheet #7 solution

1) a) $\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} e^{-4x}\,dx = \left(-\frac{1}{4}e^{-4x}\right)\Big|_0^{+\infty} = 0 - \left(-\frac{1}{4}\right) = \frac{1}{4} \neq 1$

Hence NO a pdf

b) $\sin(x)$ is negative in $[\pi, 3\pi/2]$, hence NO a pdf

c) ·) $h(x) \geq 0$ $\left(4/x^5 \geq 0 \text{ if } x \geq 1\right)$

·) $\int_{-\infty}^{+\infty} h(x)\,dx = \int_1^{+\infty} \frac{4}{x^5}\,dx = \left(-\frac{1}{x^4}\right)\Big|_1^{+\infty} = (0 - (-1)) = 1$

Hence it IS a pdf

2) a) $\lim_{x \to +\infty} e^x = +\infty \neq 1$, hence NO a cdf

b) $\lim_{x \to +\infty} e^{-x} = 0 \neq 1$ or $\lim_{x \to -\infty} e^{-x} = -\infty \neq 0$, hence NO a cdf

c) $\lim_{x \to +\infty} \frac{10}{6+4e^{-x}} = \frac{10}{6} \neq 1$

I intended this to be a Yes, so let's look at $\frac{10}{10+e^{-x}}$

$\lim_{x \to +\infty} \frac{10}{10+e^{-x}} = \frac{10}{10+0} = 1$, $\lim_{x \to -\infty} \frac{10}{10+e^{-x}} = 0$ since $e^{-x} \to +\infty$

$H'(x) = \frac{10e^{-x}}{(10+e^{-x})^2} > 0$, then H is increasing

hence $H(x)$ is a cdf

d) $\lim_{x \to -\infty} 1 = 1 \neq 0$, hence is not a cdf

3) For 1.c) $E(X) = \int_1^{+\infty} x\left(\frac{4}{x^5}\right)dx = \int_1^{+\infty} \frac{4}{x^4}\,dx = \left(\frac{-4}{3x^3}\right)\Big|_1^{+\infty}$

$$= \left(0 - \left(-\frac{4}{3}\right)\right) = \frac{4}{3}$$

For 2.c) $E(X) = \int_{-\infty}^{+\infty} x \frac{10e^{-x}}{(10+e^{-x})^2} dx \ \ldots \ (*)$

$\int x \frac{10e^{-x}}{(10+e^{-x})^2} dx = x \frac{10}{10+e^{-x}} - \int \frac{10}{10+e^{-x}} dx$ (integration by parts)

Now: $\frac{10}{10+e^{-x}} = 1 - \frac{e^{-x}}{10+e^{-x}}$

So $\int \frac{10}{10+e^{-x}} dx = x - \int \frac{e^{-x}}{10+e^{-x}} dx$. By u-sub $(u = e^{-x}, du = -e^{-x} dx)$

$$= x + \int \frac{du}{10+u} = x + \ln(10+u) = x + \ln(10+e^{-x})$$

Now $(*)$ $\left[x \frac{10}{10+e^{-x}} - x - \ln(10+e^{-x})\right]\Big|_{-\infty}^{+\infty}$

$$= \left[-\frac{xe^{-x}}{10+e^{-x}} - \ln(10+e^{-x})\right]\Big|_{-\infty}^{+\infty}$$

$\lim_{x \to +\infty} \frac{-xe^{-x}}{10+e^{-x}} = 0$, $\lim_{x \to +\infty} \ln(10+e^{-x}) = \ln(10)$

$\lim_{x \to -\infty} \frac{-xe^{-x}}{10+e^{-x}} - \ln(10+e^{-x}) = \lim_{x \to -\infty} \frac{-x}{10e^{x}+1} - \ln(e^{-x} \cdot (10e^{x}+1))$

$= \lim_{x \to -\infty} \frac{-x}{10e^{x}+1} + x - \ln(10e^{x}+1) = \lim_{x \to -\infty} \frac{x\,10e^{x}}{10e^{x}+1} - \ln(10e^{x}+1)$

$$= \frac{0}{0+1} - \ln(0+1) = 0$$

$\Rightarrow E(X) = 0 + \ln(10) - 0 = \ln(10)$